

JAMES BOUNDARIES AND σ -FRAGMENTED SELECTORS

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ABSTRACT. We study the boundary structure for w^* -compact subsets of dual Banach spaces. Being more precise, for a Banach space X , $0 < \varepsilon < 1$ and T a subset of the dual space X^* such that $\bigcup\{B(t, \varepsilon) : t \in T\}$ contains a James boundary for B_{X^*} we study different kind of conditions on T , besides T being countable, which ensure that

$$(SP) \quad X^* = \overline{\text{span } T}^{\|\cdot\|}.$$

We analyze two different non separable cases where the equality (SP) holds: (a) if $J : X \rightarrow 2^{B_{X^*}}$ is the duality mapping and there exists a σ -fragmented map $f : X \rightarrow X^*$ such that $B(f(x), \varepsilon) \cap J(x) \neq \emptyset$ for every $x \in X$, then (SP) holds for $T = f(X)$ and in this case X is Asplund; (b) if T is weakly countably K -determined then (SP) holds, X^* is weakly countably K -determined and moreover for every James boundary B of B_{X^*} we have $B_{X^*} = \overline{\text{co}(B)}^{\|\cdot\|}$. Both approaches use Simons' inequality and ideas exploited by Godefroy in the separable case (*i.e.*, when T is countable). While proving (a) we prove that X is Asplund if, and only if, the duality mapping has an ε -selector, $0 < \varepsilon < 1$, that sends separable sets into separable ones. A consequence of the above is that the dual unit ball B_{X^*} is norm fragmented if, and only if, it is norm ε -fragmented for some fixed $0 < \varepsilon < 1$. Our analysis is completed by offering a characterization of those Banach spaces (non necessarily separable) without copies of ℓ^1 via the structure of the boundaries of w^* -compact sets of their duals. Several applications and complementary results are proved. Our results extend to the non separable case results by Godefroy, Contreras-Payá and Rodé.

1. INTRODUCTION

Given a Banach space X and a w^* -compact subset $K \subset X^*$, a James boundary for K is a subset B of K such that for every $x \in X$ there exists some $b \in B$ such that $b(x) = \sup\{k(x) : k \in K\}$. If K is moreover convex the classical James boundary $\text{Ext } K$ of the set of the extreme points of K allows us to recover K through the equality $K = \overline{\text{co}(\text{Ext } K)}^{w^*}$. In general, James boundaries can be even disjoint of the set of extreme points. Therefore the mere idea of studying how properties of a given boundary are reflected on K has been a research field of continuous interest with applications to the theory of general Banach spaces, optimization, Fourier analysis, etc. Here is a non exhaustive list of papers and books dealing with these kind of problems [4, 5, 9, 12, 15, 16, 22, 23, 26, 28, 38, 40, 47]: the reference [13] offers an excellent survey about infinite dimensional convexity and in particular about integral representation theorems and boundaries.

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Our starting point is the following result:

Theorem 1.1 ([4, 13, 16]). *Let X be a Banach space, $0 < \varepsilon < 1$ and T a countable subset of X^* such that $\bigcup\{B(t, \varepsilon) : t \in T\}$ contains a James boundary for B_{X^*} . Then $X^* = \overline{\text{span } T}^{\|\cdot\|}$ and therefore X^* is separable.*

Note that under the conditions in Theorem 1.1, once we know that X^* is separable a result by Rodé, see [40], can be used to obtain that for every James boundary B of B_{X^*} we have $B_{X^*} = \overline{\text{co}(B)}^{\|\cdot\|_{X^*}}$: using Fonf and Lindenstrauss' terminology, [12], when the last equality holds we will say that B has property (S) –(S) stands for *strong*.

We aim in this paper to extend the results above to the non separable case (T will be then non countable) and thus answer a question raised in [38]. We envisage two different ways of extending the previous results to the non separable case:

✓ *Using ε -selectors for the duality mapping.* If $(X, \|\cdot\|)$ is a Banach space the duality mapping $J : X \rightarrow 2^{B_{X^*}}$ is defined at each $x \in X$ by

$$J(x) := \{x^* \in B_{X^*} : x^*(x) = \|x\|\}.$$

Our main result here, Theorem 4.1, states that if $f : X \rightarrow X^*$ is a σ -fragmented map such that $B(f(x), \varepsilon) \cap J(x) \neq \emptyset$ for every $x \in X$ then $X^* = \overline{\text{span } f(X)}^{\|\cdot\|}$ and in this case X is Asplund.

✓ *Using descriptive properties of T .* Our result here, Theorem 5.1, says that if $T \subset X^*$ is weakly countably K -determined and there exists $0 < \varepsilon < 1$ such that $\bigcup\{B(t, \varepsilon) : t \in T\}$ contains a James boundary for B_{X^*} , then $X^* = \overline{\text{span } T}^{\|\cdot\|}$ is weakly countably K -determined and every James boundary B of B_{X^*} has property (S).

Since every separable metric space is countably K -determined our second approach clearly extends Theorem 1.1 and its consequence above. The notion of σ -fragmented map, see Definition 1, is truly wide and in particular each function with countable range is σ -fragmented: thus our first approach also extends Theorem 1.1.

A brief description of the contents of the paper follows. Section 2 is devoted to the study of the notion of σ -fragmentability for single-valued and set-valued maps. We obtain a characterization of set-valued σ -fragmented maps via ε -selectors that are either piecewise barely constant or piecewise barely continuous, Theorem 2.1. σ -fragmented maps are precisely the uniform limits of piecewise barely constants maps, Corollary 2.2; pointwise cluster points of σ -fragmented maps are σ -fragmented, Proposition 2.3. The relationship between σ -fragmentability and networks is stated in one of the key results in this paper, Theorem 2.5, that leads to the other key result Theorem 2.8: the last result states how σ -fragmented maps send separable sets into separable ones.

In Section 3 we specialize the results of the previous section for the identity map from a Banach space equipped with its weak topology into itself with the norm metric. By doing so we exhibit several properties of σ -fragmented Banach spaces following the scheme presented in [31] for the renorming case.

In Section 4 we prove one of our main results already commented, Theorem 4.1. We also characterize Asplund spaces as those Banach spaces X for which the duality mapping J has an ε -selector, $0 < \varepsilon < 1$, $f : X \rightarrow X^*$ that sends separable

subsets of X into separable subsets of X^* , Theorem 4.2 and Corollary 4.3. A consequence of the above is that the dual unit ball B_{X^*} is norm fragmented if, and only if, it is norm ε -fragmented for some fixed $0 < \varepsilon < 1$, Corollary 4.6. The section is closed with Theorem 4.7 and its Corollary where results as above are proved for any w^* -compact subset of X^* .

Section 5 starts with the proof of our Theorem 5.1 already presented above. Theorem 5.4 offers a characterization of those Banach spaces (non necessarily separable) without copies of ℓ^1 via the structure of the boundaries of w^* -compact sets of their duals and the topology γ on X^* of uniform convergence on bounded and countable sets of X . The paper is finished by giving our proof in Corollary 5.6 of the fact that for a dual Banach space X^* with property \mathcal{C} all boundaries for B_{X^*} have property (S).

A bit of terminology: Most of our notation and terminology is standard otherwise it is either explained here or whenever it is needed: unexplained concepts and terminology can be found in our standard references for Banach spaces [5, 9] and topology [7, 29]. By letters T, E, X, \dots we denote topological spaces here. Sometimes our topological spaces are assumed to be metric and then the letters d, ρ, \dots denote the metrics on them. If (E, ρ) is a metric space, $x \in E$ and $\delta > 0$ we denote by $B_\rho(x, \delta)$ (or $B(x, \delta)$ if no confusion arises) the open ρ -ball centered at x of radius δ ; if $A \subset E$ we write

$$\text{diam}(A) := \sup\{d(x, y) : x, y \in A\}.$$

All our vector spaces E, X, \dots are assumed to be real. Sometimes E is assumed to be a normed space with the norm $\|\cdot\|$: the letter X is reserved to denote a Banach space. Given a subset S of a vector space, we write $\text{co}(S)$, $\text{aco}(S)$ and $\text{span}(S)$ to denote, respectively, the convex, absolutely convex and linear hull of S . In the normed space $(E, \|\cdot\|)$ the unit ball $\{x \in E : \|x\| \leq 1\}$ is denoted by B_E . Thus the unit ball of E^* is B_{E^*} . If S is a subset of E^* , then $\sigma(E, S)$ denotes the weakest topology for X that makes each member of S continuous, or equivalently, the topology of pointwise convergence on S . Dually, if S is a subset of E , then $\sigma(E^*, S)$ is the topology for E^* of pointwise convergence on S . In particular $\sigma(E, E^*)$ and $\sigma(E^*, E)$ are the weak (w) and weak* (w^*) topologies respectively. Of course, $\sigma(X, S)$ is always a locally convex topology and it is Hausdorff if and only if $X^* = \overline{\text{span } S}^{w^*}$ and similarly for $\sigma(X^*, S)$. Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x \rangle$ and $x^*(x)$ to denote the evaluation of x^* at x .

2. σ -FRAGMENTED MAPS

Our main tool to face the problems presented in the introduction is the notion of σ -fragmented map that was introduced in [26] in order to deal with selection problems. Since its introduction this notion has been used in different settings by different authors as for instance in [35]. In this section we will present a detailed study of σ -fragmented maps which is close in spirit to the properties studied for σ -continuous maps in connection with renorming properties of Banach spaces in [32].

Definition 1 ([26]). *Let f be a map from a topological space (T, τ) into a metric space (E, ρ) . Let S be a subset of T . We say that $f|_S$ is ρ -fragmented down to ε or ε -fragmented for some $\varepsilon > 0$, if whenever C is a non-empty subset of S , there exists a τ -open subset V in T such that $C \cap V \neq \emptyset$ and $\rho - \text{diam}(f(C \cap V)) < \varepsilon$:*

we simply use *fragmented* instead of ρ -fragmented when ρ is understood. Given $\varepsilon > 0$ we say that f is ε - σ -fragmented if there exists countable family of subsets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ that covers T such that $f|_{T_n^\varepsilon}$ is ε -fragmented for every $n \in \mathbb{N}$.

The map f is said to be σ -fragmented if it is ε - σ -fragmented for each $\varepsilon > 0$.

For set-valued maps the corresponding notion of σ -fragmentability is below:

Definition 2 ([26]). Let F be a set-valued map from a topological space (T, τ) into the subsets of a metric space (E, ρ) . Let S be a subset of T . We say that $F|_S$ is fragmented down to ε for some $\varepsilon > 0$, if whenever C is a non-empty subset of S , there exists a τ -open subset V of S with $V \cap C \neq \emptyset$ and a subset D of E with $\rho - \text{diam}(D) < \varepsilon$ such that $F(t) \cap D \neq \emptyset$ for every $t \in V \cap C$. Given $\varepsilon > 0$ we say that F is ε - σ -fragmented if there exists countable family of subsets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ that covers T such that $F|_{T_n^\varepsilon}$ is ε -fragmented for every $n \in \mathbb{N}$.

The set-valued map F is said to be σ -fragmented if it is ε - σ -fragmented for each $\varepsilon > 0$.

It is easily proved that in the above definitions of ε -fragmentability for f and F the sets “ C ’s” can be taken to be closed without loss of generality. The easiest but non trivial examples of σ -fragmented maps are provided by the class of maps introduced in the following definition: we refer to [30] for the concept of barely continuous function.

Definition 3. A map f from a topological space (T, τ) into a metric space (E, ρ) is said to be barely continuous, (resp. barely constant) if for every non-empty closed set $A \subset T$ the restriction $f|_A$ has at least one-point of continuity (resp. there exists an τ -open set $W \subset T$ such that $W \cap A \neq \emptyset$ and $f|_{A \cap W}$ is constant).

We say that f is piecewise barely continuous (resp. piecewise barely constant) if there exists a countable family of subsets $\{T_n : n \in \mathbb{N}\}$ that covers T such that $f|_{T_n}$ is barely continuous (resp. barely constant) for each $n \in \mathbb{N}$.

Baire’s Great Theorem establishes that a map f from a complete metric space T into a Banach space E is barely continuous if, and only if, f is the pointwise limit of a sequence of continuous functions, *i.e.*, f is a Baire one map, see [5, Theorem I.4.1]. It was proved in [26, Corollary 7] that a map f from a perfectly paracompact space T into a Banach space E is σ -fragmented with closed sets T_n^ε in Definition 1 if, and only if, it is a Baire one map. Corollary 7 in [26] is based upon the approximation result [26, Theorem 5] that is established there for σ -fragmented maps by closed sets T_n^ε : if we drop off the closedness of the T_n^ε ’s and only care about σ -fragmentability we can prove the following:

Theorem 2.1. Let F be a set-valued map from a topological space (T, τ) into the subsets of a metric space (E, ρ) . The following statements are equivalent:

- (i) F is σ -fragmented;
- (ii) for every $\varepsilon > 0$ there exists a piecewise barely constant map $f_\varepsilon : T \rightarrow E$ such that $\rho - \text{dist}(f_\varepsilon(t), F(t)) < \varepsilon$ for every $t \in T$.
- (iii) for every $\varepsilon > 0$ there exists a piecewise barely continuous map $f_\varepsilon : T \rightarrow E$ such that $\rho - \text{dist}(f_\varepsilon(t), F(t)) < \varepsilon$ for every $t \in T$.

Proof. (i) \Rightarrow (ii) Fix $\varepsilon > 0$. According to Definition 1 let us decompose T as $T = \bigcup_{n=1}^{\infty} T_n^\varepsilon$ in such a way that for each non-empty subset C of T_n^ε there exists an open

subset V of T and a subset D of E with $\rho - \text{diam}(D) < \varepsilon$, such that $V \cap C \neq \emptyset$ and

$$F(t) \cap D \neq \emptyset \quad \text{for every } t \in V \cap C.$$

Without any loss of generality we can assume that the sets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ are pairwise disjoint: now we will construct for every n a barely constant function $f_n^\varepsilon : T_n^\varepsilon \rightarrow E$ with

$$\rho - \text{dist}(f_n^\varepsilon(t), F(t)) < \varepsilon \quad \text{for every } t \in T_n^\varepsilon.$$

Once the above has been proved, statement (ii) is satisfied if we define

$$f_\varepsilon(t) := f_n^\varepsilon(t) \text{ for } t \in T_n^\varepsilon, n \in \mathbb{N}.$$

Let us fix $n \in \mathbb{N}$ and let us construct f_n^ε . Since $F|_{T_n^\varepsilon}$ is ε -fragmented, an argument of transfinite induction provide us with well ordered families of open subsets $\{G_\gamma : \gamma < \Gamma_\varepsilon^n\}$ of T_n^ε together with subsets $\{D_\gamma : \gamma < \Gamma_\varepsilon^n\}$ of E with $\rho - \text{diam}(D_\gamma) < \varepsilon$, $\gamma < \Gamma_\varepsilon^n$, such that for each $\mu < \Gamma_\varepsilon^n$ we have

$$M_\mu := G_\mu \setminus \left(\bigcup \{G_\gamma : \gamma < \mu\} \right) \neq \emptyset \text{ and } F(t) \cap D_\mu \neq \emptyset \text{ for every } t \in M_\mu$$

and

$$T_n^\varepsilon = \bigcup \{G_\gamma : \gamma < \Gamma_\varepsilon^n\}.$$

For each $\gamma < \Gamma_\varepsilon^n$ we pick a point y_γ in D_γ and define $f_n^\varepsilon(t) := y_\gamma$ whenever $t \in M_\gamma$. The map $f_n^\varepsilon : T_n^\varepsilon \rightarrow E$ is barely constant. Indeed, if for a non-empty subset A of T_n^ε we define γ_0 to be the first ordinal with $A \cap G_{\gamma_0} \neq \emptyset$ then we have that f_n^ε is constant on $A \cap G_{\gamma_0}$ because

$$A \cap G_{\gamma_0} \subset M_{\gamma_0} = G_{\gamma_0} \setminus \bigcup \{G_\beta : \beta < \gamma_0\}$$

and $f_n^\varepsilon(t) = y_{\gamma_0}$ for every $t \in M_{\gamma_0}$. On the other hand, since $\gamma < \Gamma_\varepsilon^n$ and $t \in M_\gamma$ imply that $f_n^\varepsilon(t) = y_\gamma$ with

$$y_\gamma \in D_\gamma, \rho - \text{diam}(D_\gamma) < \varepsilon \text{ and } F(t) \cap D_\gamma \neq \emptyset,$$

we conclude that

$$\rho - \text{dist}(f_n^\varepsilon(t), F(t)) < \varepsilon \text{ for every } t \in T_n^\varepsilon,$$

and the proof is over.

Being the implication (ii) \Rightarrow (iii) obvious we prove (iii) \Rightarrow (i). Fix $\varepsilon > 0$ and take $f_\varepsilon : T \rightarrow E$ such that we can decompose $T = \bigcup \{T_n^\varepsilon : n \in \mathbb{N}\}$ and $f_\varepsilon|_{T_n^\varepsilon}$ is barely continuous for each $n \in \mathbb{N}$ with

$$\rho - \text{dist}(f_\varepsilon(t), F(t)) < \frac{\varepsilon}{3} \text{ for every } t \in T.$$

If C is a non-empty closed subset of T_n^ε , then there exists an open set V of T such that $V \cap C \neq \emptyset$ and $\rho - \text{diam}(f_\varepsilon(V \cap C)) < \frac{\varepsilon}{3}$. If we define

$$D := \{y \in E : \rho - \text{dist}(y, f_\varepsilon(V \cap C)) < \frac{\varepsilon}{3}\}$$

then we have that $\rho - \text{diam}(D) < \varepsilon$ and $F(t) \cap D \neq \emptyset$ for every $t \in V \cap C$. The proof is over. \square

When we deal with a single valued map the above result is the key to prove that the barely constant maps together with countable splitting and limit points produce all σ -fragmented maps.

Corollary 2.2. *A map f from a topological space (T, τ) into a metric space (E, ρ) is σ -fragmented if, and only if, there exists a sequence $\{f_n : T \rightarrow E, n \in \mathbb{N}\}$, of piecewise barely constant maps that uniformly converges to f .*

Proof. By Theorem 2.1 the map $f : T \rightarrow E$ is σ -fragmented if, and only if, we have that for $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, there exists a piecewise barely constant map $f_n : T \rightarrow E$ with $\rho(f_n(t), f(t)) < \frac{1}{n}$ for all $t \in T$. \square

Next we show that σ -fragmentability is in fact preserved when taking pointwise cluster points of sequences of σ -fragmented maps:

Proposition 2.3. *Let f be a map from a topological space (T, τ) into a metric space (E, ρ) . If there exists a sequence $\{f_n : T \rightarrow E : n = 1, 2, \dots\}$ of σ -fragmented maps with*

$$f(t) \in \overline{\{f_n(t) : n = 1, 2, \dots\}}^\rho \text{ for every } t \in T,$$

then f is σ -fragmented.

Proof. Let us fix $\varepsilon > 0$ and let us define the sets

$$S_n^\varepsilon := \{t \in T : \rho(f(t), f_n(t)) < \frac{\varepsilon}{3}\} \quad n = 1, 2, \dots$$

Clearly $T = \bigcup_{n=1}^{\infty} S_n^\varepsilon$ and for every n we can also decompose $T = \bigcup_{m=1}^{\infty} T_m^{n,\varepsilon}$ in such a way that $f_n|_{T_m^{n,\varepsilon}}$ is $\frac{\varepsilon}{3}$ -fragmented for every $m = 1, 2, \dots$. Observe that we have

$$S_n^\varepsilon = \bigcup_{m=1}^{\infty} S_n^\varepsilon \cap T_m^{n,\varepsilon} \text{ and } T = \bigcup_{n,m=1}^{\infty} S_n^\varepsilon \cap T_m^{n,\varepsilon}.$$

Now, for every pair of $n, m \in \mathbb{N}$ if we take a non-empty subset C of $S_n^\varepsilon \cap T_m^{n,\varepsilon}$, then there exists a non-empty open set V of T with $V \cap C \neq \emptyset$ and $\rho - \text{diam}(f_n(V \cap C)) < \frac{\varepsilon}{3}$; the last inequality and the fact that $C \subset S_n^\varepsilon$ leads to

$$\rho - \text{diam}(f(V \cap C)) < \varepsilon,$$

when bearing in mind the definition of S_n^ε . The proof is over. \square

For maps with values in a normed space $(E, \|\cdot\|)$ the term σ -fragmented will always refer to σ -fragmentability with respect to the given norm $\|\cdot\|$ in E .

Corollary 2.4. *Let f be a map from a topological space (T, τ) into the normed space $(E, \|\cdot\|)$. If there exists a sequence $\{f_n : T \rightarrow E : n = 1, 2, \dots\}$ of σ -fragmented maps with*

$$f(t) \in \overline{\{f_n(t) : n = 1, 2, \dots\}}^w \text{ for every } t \in T,$$

then f is σ -fragmented.

Proof. It is easily checked that linear combinations of σ -fragmented maps are σ -fragmented. Hence the \mathbb{Q} -linear combinations of $\{f_n : n = 1, 2, \dots\}$ is a countable family $\{g_n : T \rightarrow E : n = 1, 2, \dots\}$ of σ -fragmented maps for which Hahn-Banach theorem, [9, Theorem 3.19], gives us that

$$f(t) \in \overline{\{g_n(t) : n = 1, 2, \dots\}}^{\|\cdot\|} \text{ for every } t \in T.$$

We apply now Proposition 2.3 and the proof is finished. \square

The following definition can be found in [20].

Definition 4 (Hansell, [20]). *A family of subsets \mathcal{E} in a topological space T is said to be scattered if \mathcal{E} is disjoint and there exists a well ordering \leq of \mathcal{E} such that for every $E \in \mathcal{E}$ the set $\cup\{M \in \mathcal{E} : M \leq E\}$ is open relative to $\cup\mathcal{E}$. A family of subsets \mathcal{E} is said to be σ -scattered if it can be decomposed into a countable union $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ such that every family \mathcal{E}_n is scattered.*

Given a map $f : T \rightarrow E$ between topological spaces, we say that a family \mathcal{B} of subsets of T is a *function base* for f if, whenever V is open in E , then $f^{-1}(V)$ is union of sets of \mathcal{B} , i.e., \mathcal{B} is a function base for f if it is a *network* in T for the topology given by $\{f^{-1}(V) : V \text{ is open in } E\}$, see [7, p. 170] for the notion of network. We recall that a family $\{F_i : i \in I\}$ in a topological space (T, τ) is said to be discrete if for every point $x \in T$ there exists a neighbourhood U of x such that U meets at most one member of the family $\{F_i : i \in I\}$, [7, p. 33]. Recall also that a family of subsets $\{D_j : j \in J\}$ of T is a refinement of the family $\{C_l : l \in L\}$ if $\bigcup_{j \in J} D_j = \bigcup_{l \in L} C_l$ and each D_j is contained in some C_l , [7, p. 165].

We note that next result already appeared in [20, Theorem 1.10] for the very particular case of f being the identity map $id : (T, \tau) \rightarrow (T, \rho)$ where ρ is a metric on T whose associated topology is finer than τ . Next theorem exhibits the relationship between the notion of σ -fragmented map and the previous concept of map with σ -scattered function base introduced by Hansell, see [19] and the references therein.

Theorem 2.5. *Let f be a map from a topological space (T, τ) into the metric space (E, ρ) . The following statements are equivalent:*

- (i) f is σ -fragmented;
- (ii) if $\{D_i : i \in I\}$ is a discrete family of subsets in (E, ρ) then the family $\{f^{-1}(D_i) : i \in I\}$ has a σ -scattered refinement;
- (iii) f has a σ -scattered function base.

Proof. (i) \Rightarrow (ii) Let $\{D_i : i \in I\}$ be a discrete family in (E, ρ) and let us define

$$D_{i,p} := \{x \in D_i : B_\rho(x, 1/p) \cap D_j = \emptyset \text{ for each } j \in I, j \neq i\}$$

for every positive integer p . We clearly have $D_i = \bigcup_{p=1}^{\infty} D_{i,p}$ for every $i \in I$ and the family $\{D_{i,p} : i \in I\}$ is $1/p$ -discrete –meaning that the distance between two different elements of the family is at least $1/p$, for every $p = 1, 2, \dots$. Fix the positive integer p and let us use that f is $1/p$ - σ -fragmented to produce the decomposition $T = \bigcup_{n=1}^{\infty} T_n^{1/p}$ such that, for every n we have a well ordered family of relatively open subsets $\{G_\gamma^{n,p} : \gamma < \Gamma_n^{1/p}\}$ on $T_n^{1/p}$ which covers $T_n^{1/p}$ and provide us with the scattered family

$$\{M_\mu^{n,p} := G_\mu^{n,p} \setminus \bigcup\{G_\beta^{n,p} : \beta < \mu\} \text{ for } \mu < \Gamma_n^{1/p}\}$$

such that

$$\rho - \text{diam}(f(M_\mu^{n,p})) < \frac{1}{p} \text{ for every } \mu < \Gamma_n^{1/p}.$$

The $1/p$ -discreteness of the family $\{D_{i,p} : i \in I\}$ implies that the set $M_\mu^{n,p}$ meets at most one member of the family $\{f^{-1}(D_{i,p}) : i \in I\}$. Thus the family formed by all the non void sets of the form

$$\{M_\mu^{n,p} \cap f^{-1}(D_{i,p})\},$$

for $i \in I$ and $\mu < \Gamma_{n,p}^{1/p}$ is scattered for fixed integers n and p because any subset of ordinals is a well ordered set.

All things considered we conclude that

$$\bigcup_{n,p=1}^{\infty} \{M_{\mu}^{n,p} \cap f^{-1}(D_{i,p}) : i \in I, \mu < \Gamma_n^{1/p}\}$$

is a σ -scattered refinement of the family $\{f^{-1}(D_i) : i \in I\}$ and (ii) is fulfilled.

(ii) \Rightarrow (iii) Stone's theorem, [7, Theorem 4.4.3], provides us with a σ -discrete base \mathcal{B} for the metric topology in (E, ρ) , that is, \mathcal{B} is base for the topology and it can be split as $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ with each \mathcal{B}_n is discrete. According to (ii) each $f^{-1}(\mathcal{B}_n)$ has a σ -scattered refinement, that is, there exist scattered families \mathcal{E}_m^n in T with $\bigcup_{m=1}^{\infty} \mathcal{E}_m^n$ being a refinement of $f^{-1}(\mathcal{B}_n)$. Observe that $\bigcup_{n,m=1}^{\infty} \mathcal{E}_m^n$ is a σ -scattered refinement of $f^{-1}(\mathcal{B})$. Furthermore, we claim that $\bigcup_{n,m=1}^{\infty} \mathcal{E}_m^n$ is a function base of f . Indeed, given an open set $V \subset E$ and $x \in f^{-1}(V)$ there exists B in some \mathcal{B}_n such that $x \in f^{-1}(B) \subset f^{-1}(V)$; from the equality

$$\bigcup_{B \in \mathcal{B}_n} f^{-1}(B) = \bigcup_m \bigcup_{C \in \mathcal{E}_m^n} C$$

and the facts that $\{f^{-1}(B) : B \in \mathcal{B}_n\}$ are disjoint together with $\bigcup_{m=1}^{\infty} \mathcal{E}_m^n$ being a refinement of $f^{-1}(\mathcal{B}_n)$ we infer that there exists C in some \mathcal{E}_m^n such that

$$x \in C \subset f^{-1}(B) \subset f^{-1}(V),$$

and the proof for this implication is over.

(iii) \Rightarrow (i) Let $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ be a function base for f with \mathcal{E}_n scattered family for every $n \in \mathbb{N}$, i.e., $\mathcal{E}_n = \{E_{\alpha}^n : \alpha < \Gamma_n\}$ and $E_{\alpha}^n \subset U_{\alpha}^n \setminus \bigcup\{U_{\beta}^n : \beta < \alpha\}$ for some well ordered family of open sets $\{U_{\alpha}^n : \alpha < \Gamma_n\}$ in T . If we fix $T_n := \bigcup\{E_{\alpha}^n : \alpha < \Gamma_n\}$ and for every $\alpha < \Gamma_n$ we choose $t_{\alpha}^n \in E_{\alpha}^n$, then we can define $f_n(t) := f(t_{\alpha}^n)$ for every $t \in T_n$ when $t \in E_{\alpha}^n$. The function f_n is barely constant on T_n : if $A \subset T_n$ is non-empty and α_0 is the first ordinal with $A \cap U_{\alpha_0}^n \neq \emptyset$ then we have that

$$A \cap U_{\alpha_0}^n \cap T_n \subset E_{\alpha_0}^n \cap T_n,$$

and therefore $f_n(t) = f_n(t_{\alpha_0}^n)$ for every $t \in A \cap U_{\alpha_0}^n \cap T_n$. Now we extend f_n to the whole T by defining it as a constant function on $T \setminus T_n$ with a fix but arbitrary value on this set. Since \mathcal{E} is a function base for f , we easily obtain that

$$f(t) \in \overline{\{f_n(t) : n = 1, 2, \dots\}}$$

for every $t \in T$: indeed, given $\varepsilon > 0$ there exist a positive integer m and $\beta < \Gamma_m$, such that $t \in E_{\beta}^m \subset f^{-1}(B_{\rho}(f(t), \varepsilon))$, thus $f_m(t) = f(t_{\beta}^m) \in B_{\rho}(f(t), \varepsilon)$. Now Proposition 2.3 applies to conclude that f is σ -fragmented and the proof is over. \square

Remark 2.6. If the scattered function base for a map f can be constructed with sets which are difference of closed sets, then the map f enjoys properties which are close to measurability; we note that under this hypothesis f is in fact Borel measurable when the domain space is, for instance, a complete metric space or a Gul'ko compact, see [21, 20, 19].

We recall that a set $S \subset T$ is a Souslin- \mathcal{F} -set in the space (T, τ) if S is the result of the Souslin operation applied to closed sets of T , *i.e.*, for some collection of closed sets $\{F_{n_1, n_2, \dots, n_k} : (n_1, n_2, \dots, n_k) \in \mathbb{N}^{(\mathbb{N})}\}$, indexed in the set $\mathbb{N}^{(\mathbb{N})}$ of finite sequences of positive integers, we have

$$S = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{k=1}^{\infty} \{F_{\alpha|k}\},$$

where $\alpha|k := (n_1, n_2, \dots, n_k)$ when $\alpha = (n_1, n_2, \dots, n_k, \dots) \in \mathbb{N}^{\mathbb{N}}$. Every Borel set in a metric space is a Souslin- \mathcal{F} -set, [41, Theorem 44]. A map between metric spaces is called analytic if the preimage of every open set is a Souslin- \mathcal{F} -set. Bearing in mind the seminal results by R. Hansell in [18] we can now state here the following:

Corollary 2.7. *Every analytic (in particular every Borel measurable) map from a complete metric space (T, d) into a metric space (E, ρ) is σ -fragmented.*

Proof. Theorem 3 of R. Hansell in [18] states that every analytic map from a complete metric space into a metric space has a σ -discrete function base. Since every discrete family of sets is clearly scattered, the conclusion in the corollary straightforwardly follows from Theorem 2.5, see also [20, Lemma 5.9]. \square

An important property of Borel maps from complete metric spaces into metric spaces is that they send separable subsets of the domain into separable subsets of the range, see for instance [43, Theorem 4.3.8]. The fact that σ -fragmented maps enjoy the same property is proved in the next result whose proof has been taken from [32, Theorem 2.15], where the result has been used as an important tool for renormings in Banach spaces: we include the proof here for the sake of completeness.

Theorem 2.8. *Let (T, d) and (E, ρ) be metric spaces and $f : T \rightarrow E$ a σ -fragmented map. Then, for every $t \in T$ there exists a countable set $W_t \subset T$ such that*

$$f(t) \in \overline{\bigcup \{f(W_{t_n}) : n = 1, 2, \dots\}}^{\rho}$$

whenever $\{t_n\}$ is a sequence converging to t in (T, d) . In particular, $f(S)$ is separable whenever S is a separable subset of T .

Proof. Let $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ be the σ -scattered function base for the map f provided by Theorem 2.5, *i.e.*, $\mathcal{E}_n = \{E_{\alpha}^n : \alpha < \Gamma_n\}$ with

$$E_{\alpha}^n \subset U_{\alpha}^n \setminus \bigcup \{U_{\beta}^n : \beta < \alpha\}$$

and $\{U_{\alpha}^n : \alpha < \Gamma_n\}$ a well ordered family of open sets in T . For every $m \in \mathbb{N}$ we set

$$E_{\alpha}^{n,m} := \{t \in E_{\alpha}^n : d - \text{dist}(t, T \setminus U_{\alpha}^n) \geq \frac{1}{m}\}.$$

We observe that for $t \in E_{\alpha}^{n,m}$ we have that

$$(2.1) \quad B_d\left(t, \frac{1}{2m}\right) \cap E_{\beta}^{n,m} = \emptyset \text{ for all } \beta \neq \alpha, \beta < \Gamma_n.$$

Indeed, for such a t we have $B_d\left(t, \frac{1}{2m}\right) \subset U_{\alpha}^n$ and thus $B_d\left(t, \frac{1}{2m}\right) \cap E_{\beta}^{n,m} = \emptyset$ for $\beta > \alpha$. On the other hand, if for some $\beta < \alpha$ we assume that there exists

some $s \in B_d\left(t, \frac{1}{2m}\right) \cap E_\beta^{n,m}$ then we have that $B_d\left(s, \frac{1}{2m}\right) \subset U_\beta^n$ and in particular $t \in U_\beta^n$ that contradicts the fact that $t \in E_\alpha^{n,m} \subset E_\alpha^n \subset U_\alpha^n \setminus \bigcup\{U_\beta^n : \beta < \alpha\}$. Thus (2.1) is proved.

Let us fix n and m in \mathbb{N} and for every t in T and every $p \in \mathbb{N}$ such that $B_d(t, 1/p)$ meets at most one member of the family $\{E_\alpha^{n,m} : \alpha < \Gamma_n\}$, we choose one $t(n, m, p)$ in $B_d(t, 1/p) \cap \bigcup\{E_\alpha^{n,m} : \alpha < \Gamma_n\}$ when this intersection is non-empty. If we now define

$$W_t := \{t(n, m, p) : n, m, p \in \mathbb{N}\}$$

we claim that

$$f(t) \in \overline{\bigcup\{f(W_{t_k}) : k \in \mathbb{N}\}}^\rho \text{ whenever } \lim_{k \rightarrow \infty} d(t_k, t) = 0.$$

To prove that let us assume that $\lim_{k \rightarrow \infty} d(t_k, t) = 0$ and let us fix $\varepsilon > 0$. There exists an element of the function base $E_\alpha^n \subset \mathcal{E}_n$, $\alpha < \Gamma_n$, with $t \in E_\alpha^n$ and

$$(2.2) \quad E_\alpha^n \subset f^{-1}(B_\rho(f(t), \varepsilon)).$$

If m is a positive integer with $t \in E_\alpha^{n,m}$ we conclude that

$$B_d\left(t, \frac{1}{2m}\right) \cap E_\beta^{n,m} = \emptyset \text{ for all } \beta < \Gamma_n, \quad \beta \neq \alpha.$$

If k is a positive integer with $d(t_k, t) < \frac{1}{4m}$ then we have that

$$B_d\left(t_k, \frac{1}{4m}\right) \subset B\left(t, \frac{1}{2m}\right)$$

and that

$$t \in B_d\left(t_k, \frac{1}{4m}\right) \cap E_\alpha^{n,m}.$$

Consequently the ball $B_d\left(t_k, \frac{1}{4m}\right)$ only meets the set $E_\alpha^{n,m}$ of the family $\{E_\beta^{n,m} : \beta < \Gamma_n\}$. Thus $t_k(n, m, 4m)$ is defined and we have that

$$t_k(n, m, 4m) \in B_d\left(t_k, \frac{1}{4m}\right) \cap E_\alpha^{n,m} \subset E_\alpha^n,$$

the inclusion (2.2) allows us to conclude that $t_k(n, m, 4m) \in f^{-1}(B_\rho(f(t), \varepsilon))$. Thus, we see that $t_k(n, m, 4m) \in W_{t_k}$ and $\rho(f(t), f(t_k(n, m, 4m))) < \varepsilon$ and the proof is over. \square

The σ -fragmentability property of maps is not only preserved by countable partitions. Indeed, if a map f is σ -fragmented when restricted to the sets of a scattered partition, then f is σ -fragmented. Being more precise we have the following result.

Proposition 2.9. *Let f be a map from a topological space (T, τ) into the metric space (E, ρ) . If there exists a well ordered family of open sets $\{G_\gamma : \gamma < \Gamma\}$ covering T such that f is σ -fragmented restricted to every atom*

$$M_\gamma = G_\gamma \setminus \bigcup\{G_\beta : \beta < \gamma\},$$

then the map f is σ -fragmented on the whole of T .

Proof. Let us fix $\varepsilon > 0$ and let us split M_γ as

$$M_\gamma = \bigcup \{M_{\gamma,\varepsilon}^n : n = 1, 2, \dots\},$$

in such a way that $f|_{M_{\gamma,\varepsilon}^n}$ is ε -fragmented. If we set $T_\varepsilon^n := \bigcup \{M_{\gamma,\varepsilon}^n : \gamma < \Gamma\}$, then it is clear that $T = \bigcup \{T_\varepsilon^n : n = 1, 2, \dots\}$. We prove now that $f|_{T_\varepsilon^n}$ is ε -fragmented: if we fix some non-empty set $C \subset T_\varepsilon^n$ and we choose γ_0 to be the first ordinal such that $C \cap M_{\gamma_0}$ is non-empty, then

$$C \cap G_{\gamma_0} = C \cap (G_{\gamma_0} \setminus \bigcup \{G_\beta : \beta < \gamma_0\}) = C \cap M_{\gamma_0}.$$

On the other hand, since $C \subseteq T_\varepsilon^n$ we have that $\emptyset \neq C \cap M_{\gamma_0} = C \cap M_{\gamma_0,\varepsilon}^n$ and therefore the ε -fragmentability of $f|_{M_{\gamma_0,\varepsilon}^n}$ applies to provide us with an open set W in (T, τ) such that $W \cap C \cap M_{\gamma_0,\varepsilon}^n$ is non-empty and

$$\rho - \text{diam}(f(W \cap C \cap M_{\gamma_0,\varepsilon}^n)) < \varepsilon.$$

Since $W \cap C \cap G_{\gamma_0} = W \cap C \cap M_{\gamma_0,\varepsilon}^n$ we have that

$$\rho - \text{diam}(f((W \cap G_{\gamma_0}) \cap C)) < \varepsilon,$$

and therefore $f|_{T_\varepsilon^n}$ is ε -fragmented. The proof is over. \square

The previous results leads us to the following one: see [25, Theorem 4.1] for the identity map. We use the following terminology: a subset A of a metric space (E, ρ) is said to be ε -separable ($\varepsilon > 0$) if there exists some countable subset H in E such that

$$A \subset \bigcup \{B_\rho(h, \varepsilon) : h \in H\}.$$

Proposition 2.10. *Let f be a map from a topological space (T, τ) into the metric space (E, ρ) . If for every $\varepsilon > 0$ there exists a countable family of subsets $\{T_n^\varepsilon : n \in \mathbb{N}\}$ that covers T such that for every $n \in \mathbb{N}$ and every non-empty subset $C \subset T_n^\varepsilon$ there exists a τ -open subset V in T with $V \cap C$ non-empty and with $f(V \cap C)$ ε -separable, then the map f is σ -fragmented.*

Proof. For a fixed $\varepsilon > 0$ we shall construct a sequence

$$\{f_n^\varepsilon : (T, \tau) \rightarrow (E, \rho) : n = 1, 2, \dots\}$$

of σ -fragmented maps such that

$$\rho - \text{dist}(f(t), \{f_n^\varepsilon(t) : n = 1, 2, \dots\}) < \varepsilon.$$

Therefore $f(t) \in \overline{\{f_n^{1/p}(t) : n, p = 1, 2, \dots\}}^\rho$ for every $t \in T$, and an appeal to Proposition 2.3 will ensure that f is σ -fragmented. For a fixed $\varepsilon > 0$ let us split $T = \bigcup_{n=1}^{\infty} T_n^\varepsilon$ satisfying the requirements in our hypothesis for the proposition. It is easily proved that in every T_n^ε we can produce a well ordered family of open sets $\{G_\gamma^n : \gamma < \Gamma_{n,\varepsilon}\}$ such that

$$\bigcup \{G_\gamma^n : \gamma < \Gamma_{n,\varepsilon}\} = T_n^\varepsilon$$

and for every atom

$$M_\gamma^n = G_\gamma^n \setminus \bigcup \{G_\beta^n : \beta < \gamma\}$$

there exists a countable set H_γ^n in E such that

$$f(M_\gamma^n) \subset \bigcup \{B_\rho(h, \varepsilon) : h \in H_\gamma^n\}.$$

If $H_\gamma^n = \{h_\gamma^n(j) : j = 1, 2, \dots\}$ we define now the maps $f_{n,j}^\varepsilon : T_n^\varepsilon \rightarrow E$ given by

$$f_{n,j}^\varepsilon(t) := h_\gamma^n(j) \text{ if } t \in M_\gamma^n.$$

Since $f_{n,j}^\varepsilon$ is constant on the atoms of a well ordered family of open sets in T_n^ε , it is σ -fragmented on the whole piece T_n^ε after Proposition 2.9. We now define arbitrarily but constant the map $f_{n,j}^\varepsilon$ on $T \setminus T_n^\varepsilon$. The new $f_{n,j}^\varepsilon$ defined on the whole T is still σ -fragmented. On the other hand, it is clear that

$$\rho - \text{dist}(f(t), \{f_{n,j}^\varepsilon(t) : j, n = 1, 2, \dots\}) < \varepsilon$$

for every $t \in T$, and the proof is over. \square

We stress that the previous result for $f = id$ has been used in [24] where it is proved that $C_p(K)$ is σ -fragmented when K is a Rosenthal compacta of functions with countably many discontinuities at most.

3. σ -FRAGMENTED NORMED SPACES

Let $(E, \|\cdot\|)$ be a normed space and τ a topology on E coarser than the norm and H a subset of E . (H, τ) is said to be fragmented (resp. σ -fragmented) by the norm of E if the inclusion $i : (H, \tau) \rightarrow (E, \|\cdot\|)$ is fragmented (resp. σ -fragmented); when $\tau = w$ we simply say that E is σ -fragmented, [25]. Recall that a subspace F of E^* is said to be *norming* if the function p of X given by

$$p(x) = \sup\{|x^*(x)| : x^* \in F \cap B_{E^*}\}$$

is a norm equivalent to $\|\cdot\|$; if this is the case then $\overline{F}^{w^*} = E^*$.

Next result is the counterpart for σ -fragmentability of [31, Theorem 8] that has been proved for LUR renormings

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a normed space and $F \subset E^*$ a norming subspace. The following statements are equivalent:*

- (i) $(E, \sigma(E, F))$ is σ -fragmented by the norm;
- (ii) the identity $id : E \rightarrow E$ is the uniform limit for the norm of a sequence of maps $\{I_n : E \rightarrow E, n = 1, 2, \dots\}$ which are piecewise barely constant for the topology $\sigma(E, F)$;
- (iii) there exists a sequence of maps $\{I_n : E \rightarrow E, n = 1, 2, \dots\}$ which are piecewise barely constant for $\sigma(E, F)$ such that

$$x \in \overline{\{I_n(x) : n = 1, 2, \dots\}}^w \text{ for every } x \in E;$$

- (iv) there exists a sequence of maps $\{I_n : E \rightarrow E, n = 1, 2, \dots\}$ which are piecewise barely constant for $\sigma(E, F)$ such that

$$x \in \overline{\text{span}\{I_n(x) : n = 1, 2, \dots\}}^{\|\cdot\|} \text{ for every } x \in E.$$

Proof. The equivalence between (i) and (ii) straightforwardly follows from Corollary 2.2. Clearly (ii) \Rightarrow (iii). The rest of the proof uses ideas that already appeared in Corollary 2.4.

(iii) \Rightarrow (iv) If $x \in \overline{\{I_n(x) : n = 1, 2, \dots\}}^w$, for every $x \in E$, then the Hahn Banach theorem, [9, Theorem 3.19], implies that

$$x \in \overline{\text{span}\{I_n(x) : n = 1, 2, \dots\}}^{\|\cdot\|} \text{ for every } x \in E$$

and thus condition (iv) is satisfied.

(iv) \Rightarrow (i) If condition (iv) is fulfilled then the set of all rational linear combinations of $\{I_n : E \rightarrow E, n = 1, 2, \dots\}$ is a countable set of maps that we denote by $\{J_n : E \rightarrow E, n = 1, 2, \dots\}$, which are σ -fragmented for the $\sigma(E, F)$ -topology, and such that for every $x \in E$ we have that

$$x \in \overline{\{J_n(x) : n = 1, 2, \dots\}}^{\|\cdot\|}.$$

Proposition 2.3 applies now to allow us to conclude that (i) is satisfied and the proof is over. \square

4. σ -FRAGMENTED ε -SELECTORS FOR THE DUALITY MAPPING

This section is devoted to prove our main results in this paper. The notion below will be used repeatedly.

Definition 5. Let F be a set-valued map from a set T into the subsets of a metric space (E, ρ) and $\varepsilon > 0$. An ε -selector for F is a function $f : T \rightarrow E$ such that

$$\rho - \text{dist}(f(t), F(t)) < \varepsilon,$$

for every $t \in T$.

Each selector for F is clearly an ε -selector, for every $\varepsilon > 0$, but a given ε -selector is not necessarily a selector. Note that ε -selectors have appeared in Theorem 2.1. Sometimes ε -selectors are the first step when finding a real selector, see for instance [26, 28, 45].

Godefroy's result quoted in Theorem 1.1 can be rephrased in the following way that suites well our purposes.

Lemma 1. Let $(X, \|\cdot\|)$ be a Banach space, $J : X \rightarrow 2^{B_{X^*}}$ the duality mapping and let f be an ε -selector of J , $0 < \varepsilon < 1$. If $Z \subset X$ is a subspace such that $f(Z)$ is separable for the norm of X^* , then

$$(4.1) \quad Z^* = \overline{\text{span } f(Z)|_Z}^{\|\cdot\|_{Z^*}}$$

and consequently Z^* is norm separable.

Proof. We start by observing that if $Z \subset X$ is a subspace and we consider the restriction map $f_Z : Z \rightarrow Z^*$ given by $f_Z(z) := f(z)|_Z$, $z \in Z$, then

$$\|\cdot\|_{Z^*} - \text{dist}(f_Z(z), J_Z(z)) \leq \|\cdot\|_{X^*} - \text{dist}(f(z), J(z)) < \varepsilon,$$

for every $z \in Z$, where $J_Z : Z \rightarrow 2^{B_{Z^*}}$ denotes the duality mapping in Z given at $z \in Z$ by $J_Z(z) := \{z^* \in B_{Z^*} : \langle z, z^* \rangle = \|z\|\}$. Hence f_Z is an ε -selector for J_Z and $f_Z(Z) = f(Z)|_Z \subset Z^*$ is norm separable. Therefore to prove the lemma there is no loss of generality if we assume that $X = Z$, what we do for the sake of simplicity when writing. Take a countable subset D of $f(X)$ for which $f(X) \subset \overline{D}^{\|\cdot\|_{X^*}}$. Observe that for every $x \in X$ we can pick some $x_x^* \in J(x)$ such that $\|x_x^* - f(x)\| < \varepsilon$. Fix $\delta > 0$ such that $\varepsilon' := \varepsilon + \delta < 1$ and for each $x \in X$ take d_x in D with $\|f(x) - d_x\| < \delta$. The set $B = \{x_x^* : x \in X\}$ is a James boundary for B_{X^*} for which we have $B \subset \bigcup \{B(d, \varepsilon') : d \in D\}$. We can apply now Godefroy's result, Theorem 1.1, to obtain that

$$X^* = \overline{\text{span}\{D\}}^{\|\cdot\|_{X^*}} = \overline{\text{span}\{f(X)\}}^{\|\cdot\|_{X^*}},$$

and the proof is over. \square

Our main result below is proved using a reduction argument to separable Banach spaces similar to the one used in [26, Theorem 26] –argument that goes back to [8]. Nonetheless, we note that our situation here is more complicated than the one in [26, Theorem 26], because we deal here with maps which are only σ -fragmented instead of the Baire one maps used in [26]: to overcome the extra difficulties in our case we will make use of the precise description of how separable sets are sent into separable ones via σ -fragmented maps, see Theorem 2.8 above.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \rightarrow 2^{B_{X^*}}$ denote the duality mapping. If for some fixed $0 < \varepsilon < 1$ there exists a σ -fragmented ε -selector $f : X \rightarrow X^*$ of J , then we have that*

$$\overline{\text{span } f(X)}^{\|\cdot\|_{X^*}} = X^*.$$

Proof. Take a linear form $g \in X^*$ and let us prove that $g \in \overline{\text{span } f(X)}^{\|\cdot\|_{X^*}}$. The idea is to construct a subspace $Z \subset X$ satisfying the assumption in Lemma 1, i.e., $f(Z) \subset X^*$ norm separable, in such a way that from the condition

$$g|_Z \in \overline{\text{span}\{f(Z)|_Z\}}^{\|\cdot\|_{Z^*}} (= Z^*)$$

it follows that

$$g \in \overline{\text{span}\{f(X)\}}^{\|\cdot\|_{X^*}}.$$

Let us find Z . For every $x \in X$ we use Theorem 2.8 to pick a countable subset W_x in X such that

$$f(x) \in \overline{\{f(W_{x_n}) : n = 1, 2, \dots\}}^{\|\cdot\|_{X^*}},$$

whenever $(x_n)_n$ converges to x in the Banach space $(X, \|\cdot\|)$.

Let us choose $\{0\} \neq Z_1 \subset X$ to be a countable \mathbb{Q} -linear subspace. Define $D_1 := \bigcup\{W_x : x \in Z_1\}$ which is also a countable set and let us write

$$C_1 := \mathbb{Q} - \text{span}\{f(W_x) : x \in D_1\} := \{h_{1,j} : j \in \mathbb{N}\}.$$

We now find vectors $\{v_{1,j} : j \in \mathbb{N}\} \subset B_X$ such that

$$\langle g - h_{1,j}, v_{1,j} \rangle \geq \|g - h_{1,j}\| - 1,$$

and we finally collect the above v 's as $F_1 := \{v_{1,j} : j \in \mathbb{N}\}$.

An induction process produces, for every $n \in \mathbb{N}$, countable sets $C_n \subset X^*$, $Z_n, D_n \subset X$ and

$$F_n := \{v_{n,j} : j = 1, 2, \dots\} \subset B_X,$$

such that

- (i) each Z_n is \mathbb{Q} -linear subspace with $Z_n \cup F_n \subset Z_{n+1}$;
- (ii) $D_n := \bigcup\{W_x : x \in Z_n\}$;
- (iii) if we enumerate $C_n := \mathbb{Q} - \text{span}\{f(W_x) : x \in D_n\} := \{h_{n,j} : j \in \mathbb{N}\}$, then we have the inequalities

$$(4.2) \quad \langle g - h_{n,j}, v_{n,j} \rangle \geq \|g - h_{n,j}\| - \frac{1}{n} \text{ for every } j \in \mathbb{N}.$$

Indeed, if Z_i, D_i and F_i have been constructed for $1 \leq i \leq n$, then we define the countable sets

$$\begin{aligned} Z_{n+1} &:= \mathbb{Q} - \text{span}\{Z_n \cup F_n\}, \\ D_{n+1} &:= \bigcup\{W_x : x \in Z_{n+1}\}, \end{aligned}$$

and once we have enumerated

$$C_{n+1} := \mathbb{Q} - \text{span}\{f(W_x) : x \in D_{n+1}\} := \{h_{n+1,j} : j \in \mathbb{N}\}$$

we simply find vectors

$$F_{n+1} := \{v_{n+1,j} : j = 1, 2, \dots\}$$

satisfying the corresponding inequality (4.2).

Let us define the subspace $Z := \overline{\bigcup\{Z_n : n = 1, 2, \dots\}}^{\|\cdot\|}$. Our construction tell us that $f(\bigcup_{n=1}^{\infty} D_n) = \bigcup\{f(W_x) : x \in \bigcup_{n=1}^{\infty} Z_n\}$. Given $z \in Z$ there exists a sequence $(z_m)_m$ in $\bigcup\{Z_n : n = 1, 2, \dots\}$ such that $\lim_m z_m = z$. Hence, by the choices we have made of the sets W 's we conclude that

$$f(z) \in \overline{\bigcup\{f(W_{z_m}) : m = 1, 2, \dots\}}^{\|\cdot\|_{X^*}} \subset \overline{f(\bigcup\{D_n : n = 1, 2, \dots\})}^{\|\cdot\|_{X^*}}.$$

In other words $f(Z) \subset \overline{f(\bigcup\{D_n : n = 1, 2, \dots\})}^{\|\cdot\|_{X^*}}$ and we can apply Lemma 1 to Z to conclude that

$$g|_Z \in Z^* = \overline{\text{span}\{f(Z)|_Z\}}^{\|\cdot\|_{Z^*}} \subset \overline{\text{span} f(\bigcup\{D_n : n = 1, 2, \dots\})|_Z}^{\|\cdot\|_{Z^*}}.$$

To finish we prove that the latter implies that $g \in \overline{\text{span}\{f(X)\}}^{\|\cdot\|_{X^*}}$. Fix $\delta > 0$ and pick $h \in \text{span} f(\bigcup\{D_n : n = 1, 2, \dots\})$ such that

$$\|g|_Z - h|_Z\|_{Z^*} < \frac{\delta}{2}.$$

On one hand, since $D_j \subset D_{j+1}$, $j \in \mathbb{N}$, we can write

$$h = \sum_{i=1}^p q_i f(d_i)|_Z, \quad q_i \in \mathbb{Q}, \quad d_i \in D_n, \quad i = 1, 2, \dots, p,$$

for some $n \in \mathbb{N}$. On the other hand, since $h = \sum_{i=1}^p q_i f(d_i) \in C_n$ we have $h = h_{n,j}$ for some $j \in \mathbb{N}$ and without loss of generality we can and do assume that n is big enough to have $\delta > \frac{2}{n}$. All things considered we conclude that:

$$\begin{aligned} \|g - h\|_{X^*} &= \|g - h_{n,j}\|_{X^*} \stackrel{(4.2)}{\leq} \frac{1}{n} + \langle g - h_{n,j}, v_{n,j} \rangle \leq \\ &\frac{\delta}{2} + \langle g - h_{n,j}, v_{n,j} \rangle \stackrel{v_{n,j} \in Z}{\leq} \frac{\delta}{2} + \|(g - h_{n,j})|_Z\|_{Z^*} \\ &= \frac{\delta}{2} + \|g|_Z - h|_Z\|_{Z^*} \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Taking $\delta \rightarrow 0$ we have that $g \in \overline{\text{span} f(X)}^{\|\cdot\|_{X^*}}$ and the proof is over. \square

Another consequence of Lemma 1 is the following improvement of a result by C. Stegall: see, for instance, [5, Theorem I.5.9 and Remark I.5.11], [45, Corollary 9] and [47]. Unless otherwise explicitly stated the notions Baire one, Borel measurability, fragmentability, etc. when used for selectors $f : X \rightarrow X^*$ always refer to the norm in X and X^* .

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \rightarrow 2^{B_{X^*}}$ be the duality mapping. The following statements are equivalent:*

- (i) X is Asplund;
- (ii) J has a Baire one selector;

- (iii) J has a σ -fragmented selector;
- (iv) for some $0 < \varepsilon < 1$, J has a σ -fragmented ε -selector;
- (v) for some $0 < \varepsilon < 1$, J has an ε -selector that sends norm separable subsets of X into norm separable subsets of X^* .

Proof. For the implication (i) \Rightarrow (ii) we use that if X is Asplund, then Theorem 8 in [27] provides a Baire one selector for J . For the implication (ii) \Rightarrow (iii) refer for instance to Proposition 2.3. The implication (iii) \Rightarrow (iv) is clear and (iv) \Rightarrow (v) follows from Theorem 2.8.

To finish, if we assume that (v) holds, then Lemma 1 applies to ensure that each separable subspace of $Z \subset X$ has separable dual Z^* . Therefore (i) holds and the proof is over. \square

We note that if we simply want to prove that (i) \Rightarrow (vi) in the above theorem there exists no need to use the full power of the Jayne-Rogers theorem, [27, Theorem 8]. Indeed, if X is an Asplund space, then Proposition 11 in [26] can be used to obtain that the duality mapping $J : X \rightarrow 2^{B_{X^*}}$ is σ -fragmented as a set-valued map: to obtain now a σ -fragmented ε -selector for J (i.e., a selector as the one in (vi) in the Theorem above) we can just apply our Theorem 2.1 in this paper.

Corollary 4.3. *Let $(X, \|\cdot\|)$ be a Banach space and let $J : X \rightarrow 2^{B_{X^*}}$ be the duality mapping. The following statements are equivalent:*

- (i) X is Asplund;
- (ii) J has a Borel measurable selector;
- (iii) J has an analytic selector.

Proof. It is a consequence of Theorem 4.2 and Corollary 2.7. \square

The equivalences in the above Corollary can be obtained also from [3, Theorem A], where the techniques used comes from vector integration.

Corollary 4.4. *Let X be a separable Banach space. The following statements are equivalent:*

- (i) the norm open subsets of B_{X^*} are Souslin- \mathcal{F} -sets in (B_{X^*}, w^*) ;
- (ii) X^* is separable.

Proof. Let us prove (i) \Rightarrow (ii). Since X is separable, (B_{X^*}, w^*) is metrizable and thus the duality map $J : X \rightarrow 2^{B_{X^*}}$ has a selector $s : X \rightarrow B_{X^*}$ that is a Baire one map for the norm in X and the metrizable topology w^* in B_{X^*} , see [27]. Since the norm open sets in B_{X^*} are Souslin- \mathcal{F} -sets in (B_{X^*}, w^*) the identity map from (B_{X^*}, w^*) to $(B_{X^*}, \|\cdot\|_{X^*})$ is analytic. Hence the composition of the identity with s is analytic too, and therefore s is a selector for J that is analytic for the norm topologies and X and X^* . Corollary 4.3 applies to conclude that X^* is separable.

The implication (ii) \Rightarrow (i) follows from the fact that in dual separable spaces X^* we have the equality of Borel sets in (X, w^*) and Borel sets in $(X^*, \|\cdot\|_{X^*})$ because there exists an equivalent dual norm on X^* such that the weak* and the norm topologies coincide on the unit sphere, see [20, 36]. \square

Remark 4.5. Let $(X, \|\cdot\|)$ be a separable Banach space with non-separable dual X^* . Although the duality mapping $J : X \rightarrow 2^{B_{X^*}}$ does not have a σ -fragmented ε -selector for the norm in X^* , $0 < \varepsilon < 1$, the arguments in the proof of the previous result tell us that J admits a selector $s : X \rightarrow B_{X^*}$ that is a Baire one map for the

norm in X and the metrizable topology w^* in B_{X^*} . Furthermore, Godefroy proved that for $\delta > 0$ there exists a subset Δ of the unit sphere of X , homeomorphic to the Cantor set $\{0, 1\}^{\mathbb{N}}$, and such that

$$\|s(x) - s(y)\| > 1 - \delta$$

for every $x \neq y$ in Δ , [16]. It is unknown if the Cantor set Δ can be constructed in such a way that

$$\inf \{\|x^* - y^*\| : x^* \in J(x), y^* \in J(y)\} > 1 - \delta,$$

for every $x \neq y$ in Δ . This question is strongly related with Problem I.3 in [5] where it is asked if X^* is separable when the set of all support functionals of B_X at all points of Gâteaux differentiability of the norm of X is norm separable, see [16]. \square

We prove next that if X is a Banach space then the unit ball (B_{X^*}, w^*) is fragmented by the norm of X^* if, and only if, it is ε -fragmented by some fixed $0 < \varepsilon < 1$. To complete the picture we relate these results with Szlenk index that is recalled in the lines that follow. Given the Banach space X and the w^* -compact set $K \subset X^*$ we define for any $\varepsilon > 0$

$$K'_\varepsilon := \{g \in K : \|\cdot\| - \text{diam}(V \cap D) \geq \varepsilon, \\ \text{for all } w^* - \text{open neigh. } V \text{ of } g\}.$$

Inductively we define for an ordinal α we set

$$K_\varepsilon^{\alpha+1} := (K'_\varepsilon)^\alpha$$

and if β is a limit ordinal

$$K_\varepsilon^\beta := \bigcap_{\alpha < \beta} K_\varepsilon^\alpha.$$

The Szlenk index of a Banach space X is defined as follows. For any $\varepsilon > 0$, let

$$S_z(X, \varepsilon) = \min\{\alpha : (B_{X^*})_\varepsilon^\alpha = \emptyset\}$$

if such an ordinal exists and ∞ otherwise. Let now set

$$S_z(X) = \sup_{\varepsilon > 0} \{S_z(X, \varepsilon)\}.$$

A Banach space is Asplund if, and only if, $S_z(X) \neq \infty$. For a survey on the Szlenk index and its applications see [17].

Theorem 4.6. *The following conditions are equivalent for a Banach space X :*

- (i) X is an Asplund space;
- (ii) there exists $0 < \varepsilon < 1$ such that (B_{X^*}, w^*) is ε -fragmented, i.e., for every non-empty subset $C \subset B_{X^*}$ there exists some w^* -open set V in B_{X^*} such that $C \cap V \neq \emptyset$ and $\|\cdot\| - \text{diam}(C \cap V) < \varepsilon$.
- (iii) there exists $0 < \varepsilon < 1$ such that the duality mapping J is ε - σ -fragmented;
- (iv) there exists some $0 < \varepsilon < 1$ with $S_z(X, \varepsilon) \neq \infty$.

Proof. The implication (i) \Rightarrow (ii) is classical: see for instance [34, 44]. The implicación (ii) \Rightarrow (iii) follows from [26, Proposition 11]. To prove that (iii) \Rightarrow (i) we read again the proof of (i) \Rightarrow (ii) in Theorem 2.1 and observe that for our given ε and J we can construct a σ -fragmented map $f_\varepsilon : (X, \|\cdot\|) \rightarrow (X^*, \|\cdot\|_{X^*})$ such that $\|\cdot\| - \text{dist}(f_\varepsilon(x), J(x)) < \varepsilon$ for every $x \in X$: now Theorem 4.2 completes the proof of (iii) \Rightarrow (i). The implication (i) \Rightarrow (iv) follows from the fact that X is

Asplund if and only if $S_z(X) \neq \infty$. To finish we prove that (iv) \Rightarrow (ii). Let C be a non-empty w^* -closed subsets of B_{X^*} . Let α_0 be the first ordinal with

$$C \cap (B_{X^*} \setminus (B_{X^*})_{\varepsilon}^{\alpha_0}) \neq \emptyset.$$

By definition of the derivation process we know that α_0 must be a successor ordinal that we can write as $\alpha_0 = \beta_0 + 1$. So we have that

$$C \subset (B_{X^*})_{\varepsilon}^{\beta_0} \quad \text{but} \quad C \cap (B_{X^*} \setminus ((B_{X^*})_{\varepsilon}^{\beta_0})') \neq \emptyset.$$

Take now some x^* in the latter non empty set. Then there is a w^* -open set $V \subset X^*$ with $x^* \in V$ and such that

$$\|\cdot\| - \text{diam}(V \cap (B_{X^*})_{\varepsilon}^{\beta_0}) < \varepsilon.$$

Thus $W := (B_{X^*} \setminus ((B_{X^*})_{\varepsilon}^{\beta_0})') \cap V$ is a w^* -open set with $C \cap W \neq \emptyset$ because $x^* \in C \cap W$ and

$$\|\cdot\| - \text{diam}(C \cap W) < \varepsilon,$$

because $C \cap W \subset (B_{X^*})_{\varepsilon}^{\beta_0} \cap V$. \square

We note that results in the same spirit than ours previous Theorem have been proved in [14, 10].

When we replace B_{X^*} by an arbitrary w^* -compact subset of X^* we can still obtain some of the equivalences in Theorem 4.2 and its consequence Theorem 4.6.

Theorem 4.7. *Let X be Banach space, let K be a w^* -compact subset of X^* and $F_K : X \rightarrow 2^K$ the set-valued map defined by*

$$F_K(x) := \{k \in K : k(x) = \sup \{g(x) : g \in K\}\}.$$

The following conditions are equivalent:

- (i) F_K has a Baire one selector;
- (ii) F_K has a selector that sends norm separable subsets of X into norm separable subsets for X^* ;
- (iii) (K, w^*) is fragmented by the norm.

Proof. For the proof (iii) \Rightarrow (i) we refer to [26, 27, 28]. Being the implication (i) \Rightarrow (ii) clear we only give a proof for (ii) \Rightarrow (iii). Assume (ii) holds and fix a selector $f : X \rightarrow K$ sending norm separable subsets of X into norm separable subsets of X^* . According to a result by Namioka, see [34, Theorem 3.4], to prove that (K, w^*) is fragmented by the norm we only have to prove that for each countable set $A \subset B_X$ the restriction $K|_A$ of K to A is separable when equipped with the metric d_A of uniform convergence on A . If we set $Y := \overline{\text{span } A}^{\|\cdot\|}$, then $f(Y) \subset K$ is norm separable in X^* . If $r : X^* \rightarrow Y^*$ is the restriction map, then $r(K)$ is a w^* -compact subset of Y^* and $r(f(Y))$ is a James boundary for $r(K)$. Since $r(f(Y))$ is separable for the norm of Y^* Rode's result, see [40], applies to obtain that

$$r(K) \subset \overline{\text{co}(r(K))}^{w^*} = \overline{\text{co}(r(f(Y)))}^{\|\cdot\|_{Y^*}}.$$

Since $\overline{\text{co}(r(f(Y)))}^{\|\cdot\|_{Y^*}}$ is norm separable we obtain that $(r(K), \|\cdot\|_{Y^*})$ is separable. Since $A \subset B_Y$, we conclude that $(K|_A, d_A)$ is a continuous image of $(r(K), \|\cdot\|_{Y^*})$. Hence $(K|_A, d_A)$ is separable and the proof is over. \square

Remark 4.8. The implication (i) \Rightarrow (iii) in the Theorem above has been proved in [26, Theorem 26]; this implication has been proved again in [28, Theorem 7.1] where the extra hypothesis $\ell^1 \not\subset X$ is required and used and its necessity justified by the means of an example, see [28, Example 7.1]. We stress that the proviso $\ell^1 \not\subset X$ is unnecessary for these kind of results as our improvement (ii) \Rightarrow (iii) clearly shows once again. We note that the latter does not lead to any kind contradiction with Example 7.1 in [28] because this example is simply not correct. The bottom line in Example 7.1 is that it is claimed, and proved, that the attaining map $J : \ell^1 \rightarrow 2^{B_{\ell^\infty}}$ has a selector $f : (\ell^1, \|\cdot\|_1) \rightarrow (\ell^\infty, \|\cdot\|_\infty)$ that is Baire one. No such an f can be Baire one. Indeed, observe that if f is any given selector for J then $f(\ell^1)$ is a James boundary for B_{ℓ^∞} . It is easily checked that every James boundary for B_{ℓ^∞} must contain the set of all signs

$$D := \{(\varepsilon_n)_n : \varepsilon_n \in \{-1, 1\}, \text{ for every } n \in \mathbb{N}\}.$$

Since D is uncountable and the $\|\cdot\|_\infty$ -distance of two different elements of D is 2, we conclude that D is not $\|\cdot\|_\infty$ -separable. Hence $f(\ell^1)$, that contains D , is never $\|\cdot\|_\infty$ -separable. Since $(\ell^1, \|\cdot\|_1)$ is separable f cannot be Baire because otherwise $f(\ell^1)$ must be $\|\cdot\|_\infty$ -separable but it is not. \square

5. APPROXIMATION OF BOUNDARIES BY DESCRIPTIVE SETS

Let us remember now the combinatorial principle that lies in James compactness theorem as it was found by S. Simons [42], and described in the famous lemma:

Lemma 2 (Simons). *Let $(z_n)_n$ be a uniformly bounded sequence in $\ell^\infty(C)$ and let W be its convex hull. If B is a subset of C such that for every sequence of positive numbers $(\lambda_n)_n$ with $\sum_{n=1}^\infty \lambda_n = 1$ there exists $b \in B$ such that*

$$(5.1) \quad \sup\left\{\sum_{n=1}^\infty \lambda_n z_n(y) : y \in C\right\} = \sum_{n=1}^\infty \lambda_n z_n(b),$$

then

$$(5.2) \quad \sup_{b \in B} \{\limsup_{n \rightarrow \infty} z_n(b)\} \geq \inf_C \{\sup w : w \in W\}.$$

A topological space T is said to be *angelic* if, whenever A is a relatively countably compact subset of T , its closure \bar{A} is compact and each element of \bar{A} is a limit of a sequence in A : good references for angelic spaces are [11] and [37]. $C_p(T)$ stands for the space of real continuous functions endowed with the topology of pointwise convergence on T .

Lemma 3. *Let X be a Banach space, B a James boundary for B_{X^*} , $\varepsilon \geq 0$ and $T \subset X^*$ such that $B \subset \bigcup_{t \in T} B(t, \varepsilon)$. Assume that T has the property that for each $y^* \in X^*$ the compact subsets of $C_p(T \cup \{y^*\}, w)$ are angelic. The following statements hold:*

- (i) if $1 > \varepsilon$, then $X^* = \overline{\text{span } T}^{\|\cdot\|}$;
- (ii) if $\varepsilon = 0$, then $X^* = \overline{\text{span } T}^{\|\cdot\|}$ and $B_{X^*} = \overline{\text{co}(B)}^{\|\cdot\|}$.

Proof. Statement (ii) straightforwardly follows from Theorem I.2 in [15]. Statement (i) can be proved exactly with the same ideas of Lemma 4 in [16]. The proof is by contradiction. Fix $\varepsilon < \varepsilon' < 1$. If $\overline{\text{span } T}^{\|\cdot\|} \subsetneq X^*$, then there exist $x^{**} \in X^{**}$

with $\|x^{**}\| = 1$ and $x^{**}|_T = 0$. Take $y^* \in B_{X^*}$ such that $x^{**}(y^*) > \frac{1+\varepsilon'}{2}$. Consider the restrictions

$$B_X|_{T \cup \{y^*\}} \subset B_{X^{**}}|_{T \cup \{y^*\}} \subset C_p(T \cup \{y^*\}, w).$$

Since B_X is w^* -dense in $B_{X^{**}}$, our hypothesis ensure us of the existence of a sequence $(x_k)_k$ in B_X such that

$$(5.3) \quad \lim_k x^*(x_k) = x^{**}(x^*) = 0,$$

for every $x^* \in T$, and

$$\lim_k y^*(x_k) = x^{**}(y^*) > \frac{1 + \varepsilon'}{2}.$$

From the last inequality, we can assume without loss of generality that

$$(5.4) \quad y^*(x_k) > \frac{1 + \varepsilon'}{2} \text{ for every } k \in \mathbb{N}.$$

It follows from (5.3) and the inclusion $B \subset \bigcup_{t \in T} B(t, \varepsilon)$ that for every $b^* \in B$ we have

$$\limsup_{k \rightarrow \infty} b^*(x_k) \leq \varepsilon.$$

Simon's inequality, Lemma 2, applied to $C := B_{X^*}$, B and the sequence $(x_k)_k$ ensures the existence of $x \in \text{co}(\{x_k : k \in \mathbb{N}\})$ with $\|x\| < \varepsilon'$ but on the other hand we have that

$$\varepsilon' > \|x\| \geq y^*(x) \stackrel{(5.4)}{>} \frac{1 + \varepsilon'}{2}.$$

The inequality $\varepsilon' > \frac{1+\varepsilon'}{2}$ implies that $\varepsilon' > 1$ and we reach a contradiction that finishes the proof. \square

Recall that a topological space (T, τ) is said to be *countably K -determined* (resp. *K -analytic*) if there exists a upper-semicontinuous set-valued map $F : M \rightarrow 2^T$ for some separable metric space (resp. Polish space) M such that $F(M) = T$ and $F(m)$ is compact for each $m \in M$. Notice that this class of spaces does properly contain the classes of K -analytic and (so) the σ -compact spaces. The paper [46] is a milestone when speaking about Banach spaces which are countably K -determined when endowed with their weak topologies. The main result in [37] states that if T is a countably K -determined space then $C_p(T)$ is angelic.

Next theorem is the outcome of the previous preparation.

Theorem 5.1. *Let X be a Banach space, B a James boundary for B_{X^*} , $1 > \varepsilon \geq 0$ and $T \subset X^*$ such that $B \subset \bigcup_{t \in T} B(t, \varepsilon)$. If (T, w) is countably K -determined (resp. K -analytic) then:*

- (i) $X^* = \overline{\text{span } T}^{\|\cdot\|}$ and X^* is weakly countably K -determined (resp. weakly K -analytic).
- (ii) Every James boundary for B_{X^*} has property (S). In particular $B_{X^*} = \overline{\text{co}(B)}^{\|\cdot\|}$.

Proof. The equality $X^* = \overline{\text{span } T}^{\|\cdot\|}$ in (i) follows from Lemma 3 bearing in mind that $T \cup \{y^*\}$ is countably K -determined for every $y^* \in X^*$ and therefore the space $C_p(T \cup \{y^*\})$ is angelic, [37]. If (T, w) is countably K -determined (resp.

K -analytic) then $\overline{\text{span} T}^{\|\cdot\|}$ is again weakly countably K -determined (resp. K -analytic) by a result of [46] and thus (i) is proved. Statement (ii) follows from statement (ii) in Lemma 3 applied for $T = X^*$. \square

We stress that weakly countably K -determined spaces are weakly Lindelöf. Furthermore if X is a Banach space such that X^* is weakly Lindelöf, then X is Asplund, see [6, Proposition 1.8]. In particular, *a fortiori* the Banach spaces we deal with in Theorem 5.1 are Asplund spaces. We might think that for an Asplund space X and any boundary B of B_{X^*} we must have that $B_{X^*} = \overline{\text{co}(B)}^{\|\cdot\|}$. This is not true in general as the following example taken from [15] shows: let ω_1 be the first uncountable ordinal and let $X = C([0, \omega_1])$ be the space of continuous functions on $[0, \omega_1]$ equipped with the supremum norm. X is an Asplund space and if δ_α denotes the Dirac measure at α then the set $B := \{\pm\delta_\alpha : 0 \leq \alpha < \omega_1\}$ is a boundary for B_{X^*} for which $\delta_{\omega_1} \in B_{X^*} \setminus \overline{\text{co}(B)}^{\|\cdot\|}$. The best positive results in this setting that we include below are due to Haydon and Godefroy.

Theorem 5.2 (Haydon,[23]). *Let X be a Banach space. The following statements are equivalent:*

- (i) $\ell^1 \not\subset X$;
- (ii) for every w^* -compact convex subset C of X^* , $C = \overline{\text{co}(\text{Ext } C)}^{\|\cdot\|}$;
- (iii) for every w^* -compact subset K of X^* , $\overline{\text{co}(K)}^{w^*} = \overline{\text{co}(K)}^{\|\cdot\|}$.

Theorem 5.3 (Godefroy,[15]). *Let X be a separable Banach space. The following statements are equivalent:*

- (i) $\ell^1 \not\subset X$;
- (ii) for every w^* -compact subset K of X^* and every James boundary B of K we have $\overline{\text{co}(K)}^{w^*} = \overline{\text{co}(B)}^{\|\cdot\|}$;
- (iii) for every w^* -compact convex subset C of X^* and every James boundary B of C we have $C = \overline{\text{co}(B)}^{\|\cdot\|}$.

The example given on $C([0, \omega_1])$ after Theorem 5.1 shows that neither Theorem 5.2 holds for boundaries different from the *extreme points* –(i) \Rightarrow (ii) fails– nor Theorem 5.3 holds for all boundaries when X is not separable –(i) \Rightarrow (ii) fails–. Nonetheless, it is possible to have the best of the above two theorems for general Banach spaces and arbitrary James boundaries if we replace $\|\cdot\|$ in X^* by the topology γ of uniform convergence on bounded and countable subsets of X .

Theorem 5.4. *Let X be a Banach space. The following statement are equivalent:*

- (i) $\ell^1 \not\subset X$;
- (ii) for every w^* -compact subset K of X^* and any James boundary B of K we have $\overline{\text{co}(K)}^{w^*} = \overline{\text{co}(B)}^\gamma$;
- (iii) for every w^* -compact subset K of X^* , $\overline{\text{co}(K)}^{w^*} = \overline{\text{co}(K)}^\gamma$.

Proof. (i) \Rightarrow (ii) We have to prove that for each $\varepsilon > 0$, $x^* \in \overline{\text{co}(K)}^{w^*}$ and $D \subset B_X$ bounded and countable there exists $b^* \in \text{co}(B)$ such that

$$(5.5) \quad |x^*(d) - b^*(d)| < \varepsilon, \text{ for every } d \in D.$$

Define $Y := \overline{\text{span } D}$. Set $r : X^* \rightarrow Y^*$ the restriction map. Then $r(B)$ is a boundary for the w^* -compact set $r(K) \subset Y^*$. Since r is linear and w^* - w^* -continuous we obtain that

$$x^*|_Y = r(x^*) \in r(\overline{\text{co}(K)}^{w^*}) \subset \overline{\text{co}(r(K))}^{w^*}.$$

Since Y is separable and does not contain ℓ^1 , Theorem 5.3 applies to conclude that $\overline{\text{co}(r(K))}^{w^*} = \overline{\text{co}(r(B))}^{\|\cdot\|_Y}$. Therefore

$$x^*|_Y = r(x^*) \in \overline{\text{co}(r(B))}^{\|\cdot\|_Y}$$

that clearly implies (5.5).

The implication (ii) \Rightarrow (iii) is obvious. Our proof by contradiction for (iii) \Rightarrow (i) is almost the same, with a little extra remark, as the proof for (iii) \Rightarrow (i) in Theorem 5.2 as presented in [23, Theorem 3.3]. Assume that there exists an isomorphism onto its image $T : \ell^1 \rightarrow X$. Then the adjoint map $S := T^* : X^* \rightarrow (\ell^1)^*$ is onto. If we set $(e_n)_n$ to denote the canonical basis in ℓ^1 and we identify $(\ell^1)^* = \ell^\infty$ then S is nothing else but the map

$$\begin{aligned} S : X^* &\longrightarrow \ell^\infty \\ x^* &\longmapsto (x^*(Te_n))_n. \end{aligned}$$

Notice that S is w^* - w^* -continuous and also $\gamma\text{-}\|\cdot\|_\infty$ -continuous. Take a w^* -compact subset $C \subset \ell^\infty$ such that

$$(5.6) \quad \overline{\text{co}(C)}^{\|\cdot\|_\infty} \subsetneq \overline{\text{co}(C)}^{w^*},$$

see the proof of [23, Proposition 3.2]. We take now K a w^* -compact subset of X^* such that $S(K) = C$. Then we have that $\overline{\text{co}(K)}^\gamma \subsetneq \overline{\text{co}(K)}^{w^*}$, because otherwise the equality would imply

$$\begin{aligned} \overline{\text{co}(C)}^{w^*} &= \overline{\text{co}(S(K))}^{w^*} = \overline{S(\overline{\text{co}(K)})}^{w^*} = S(\overline{\text{co}(K)}^{w^*}) = \\ &= S(\overline{\text{co}(K)}^\gamma) \subset \overline{S(\overline{\text{co}(K)})}^{\|\cdot\|_\infty} = \overline{\text{co}(C)}^{\|\cdot\|_\infty}, \end{aligned}$$

that contradicts (5.6) and finishes the proof. \square

We note that implication (i) \Rightarrow (ii) in the last is indeed a James compactness theorem for the w^* -topology.

Corollary 5.5. *Let X be a Banach space such that $\ell^1 \not\subset X$. If $K \subset X^*$ is bounded, γ -closed, convex and for every $x \in X$ there exists some $k^* \in K$ such that*

$$k^*(x) = \sup \{y^*(x) : y^* \in K\}$$

then K is a w^ -compact subset of X^* .*

Proof. Since K is a James boundary for \overline{K}^{w^*} , Theorem 5.4 applies to yield the equalities below that implies that K is w^* -compact:

$$\overline{\overline{\text{co}(\overline{K}^{w^*})}^{w^*}} = \overline{\text{co}(K)}^\gamma = K. \quad \square$$

A Banach space X or more generally a convex subset M of X is said to have *property \mathcal{C}* (after Corson) if each collection of relatively closed convex subsets of M with empty intersection has a countable subcollection with empty intersection. If (M, w) is Lindelöf, then M has property \mathcal{C} since closed convex sets in X are also weak-closed. A good reference for property \mathcal{C} is [39].

The following Lemma easily follows from [1, Lemma 9] and also from the main result in [39].

Lemma 4. *Let X be a Banach space. If X^* has property \mathcal{C} , then γ is stronger than the weak topology of X^* .*

When X^* has property \mathcal{C} the above results lead to the following consequence.

Corollary 5.6. *Let X be a Banach space such that X^* has property \mathcal{C} . Then for every w^* -compact subset K of X^* and any James boundary B of K we have*

$$\overline{\text{co}(K)}^{w^*} = \overline{\text{co}(B)}^{\|\cdot\|}.$$

In particular every boundary for B_{X^} has property (S).*

Proof. On the one hand if X^* has property \mathcal{C} then $\ell^1 \not\subset X$. On the other hand if X^* has property \mathcal{C} , Lemma 4 implies that the dual of (X^*, γ) is X^{**} and therefore for any convex set $C \subset X^*$ we have that $\overline{C}^\gamma = \overline{C}^{\|\cdot\|}$. The Corollary now follows from Theorem 5.4. \square

Theorem 5.7. *Let X be a Banach space. Then for every w^* -compact weakly Lindelöf subset K of X^* and any James boundary B of K we have*

$$\overline{\text{co}(K)}^{w^*} = \overline{\text{co}(B)}^{\|\cdot\|}.$$

Proof. Theorem 4.5 in [2] ensures that

$$\overline{\text{co}(K)}^{w^*} = \overline{\text{co}(K)}^{\|\cdot\|_{X^*}}.$$

Now, we apply [2, Corollary 6.4] to obtain that $\overline{\text{span } K}^{\|\cdot\|_{X^*}}$ contains $\overline{\text{co}(K)}^{w^*}$ and it is a weakly Lindelöf determined Banach space, *i.e.*, its dual unit ball is Corson compact and in particular angelic when endowed with the w^* topology. Therefore, every element in the bidual unit ball $B_{X^{**}}$ is the limit in the topology of pointwise convergence on $\overline{\text{co}(K)}^{\sigma(X^*, X)}$ of a sequence in B_X . The conclusion follows now from Theorem I.2 in [15]. \square

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